

DIMENSIONAL DIFFERENCES BETWEEN FACES OF THE CONES OF NONNEGATIVE POLYNOMIALS AND SUMS OF SQUARES

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ABSTRACT. We study dimensions of the faces of the cone of nonnegative polynomials and the cone of sums of squares; we show that there are dimensional differences between corresponding faces of these cones. These dimensional gaps occur in all cases where there exist nonnegative polynomials that are not sums of squares. The gaps occur generically, they are not the product of selecting special faces of the cones. For ternary forms and quaternary quartics, we completely characterize when these differences are observed. Moreover, we provide an explicit description for these differences in the two smallest cases, in which the cone of nonnegative polynomials and the cone of sums of squares are different. Our results follow from more general results concerning the relationship between the second ordinary power and the second symbolic power of the vanishing ideal of points in projective space.

1. INTRODUCTION

Let $H_{n,2d}$ denote the set of homogeneous polynomials (forms) in n variables of degree $2d$ over \mathbb{R} and let \mathbb{RP}^{n-1} resp. \mathbb{CP}^{n-1} denote the $(n-1)$ -dimensional real resp. complex projective space. For a fixed number of variables n and degree $2d$, nonnegative polynomials and sums of squares form closed convex cones in $H_{n,2d}$. We call these cones $P_{n,2d}$ and $\Sigma_{n,2d}$, respectively, i.e.,

$$P_{n,2d} = \{p \in H_{n,2d} \mid p(x) \geq 0 \text{ for all } x \in \mathbb{RP}^{n-1}\},$$

$$\Sigma_{n,2d} = \left\{p \in P_{n,2d} \mid p(x) = \sum q_i^2 \text{ for some } q_i \in H_{n,d}\right\}.$$

The relationship between the cone of nonnegative polynomials and the cone of sums of squares has been studied since Hilbert's seminal paper in 1888 [11]. In this article, Hilbert showed that a nonnegative form in n variables of even degree $2d$ has to be a sum of squares of forms only in the following cases: the form is bivariate, i.e., $n = 2$, the form is quadratic, i.e., $2d = 2$, or the form is a ternary quartic, i.e., $n = 3$ and $2d = 4$. In all other cases, he proved existence of nonnegative polynomials that are not sums of squares. It is remarkable that Hilbert's proof was existential and not constructive, and the first explicit nonnegative polynomial not being a sum of squares was found only 70 years later by Motzkin [18, 19]. Recently, in small dimensions, several aspects of the differences between these cones, such as the structure of the dual cones and the algebraic boundaries of these cones, were investigated (see [3, 4, 5]).

Understanding the precise relationship between the cones $P_{n,2d}$ and $\Sigma_{n,2d}$ is interesting from the point of view of computational complexity in polynomial optimization and also for practical testing for nonnegativity (see, e.g., [12]). Indeed, while testing whether a polynomial is nonnegative is NP-hard already in degree 4 [7], testing whether a polynomial is a sum of squares can be reduced to a semidefinite programming problem, which can be solved efficiently [16].

Unfortunately, except for the cases of $n = 2$, the univariate case for nonhomogeneous polynomials, the case $2d = 2$ (see [2, Sections II.11 and II.12]), and, to some extent, the case of ternary quartics,

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neither the structure of these cones nor their precise relationship with each other is very well understood.

We focus on the study of faces of the cones $P_{n,2d}$ and $\Sigma_{n,2d}$, in particular on the investigation of their possible dimensions. A face F of a convex set K is called *exposed* if there exists a supporting hyperplane H such that $F = H \cap K$. It is easy to describe exposed faces of $P_{n,2d}$. Indeed, the boundary of the cone $P_{n,2d}$ consists of all the forms with at least one zero, whereas its interior consists of all strictly positive forms. In particular, a maximal proper face of $P_{n,2d}$ consists of all forms with exactly one prescribed zero [6, Chapter 4].

Let Γ be a set of distinct points in \mathbb{RP}^{n-1} . The forms in $P_{n,2d}$ vanishing at all points of Γ form an exposed face of $P_{n,2d}$, which we call $P_{n,2d}(\Gamma)$:

$$P_{n,2d}(\Gamma) = \{p \in P_{n,2d} \mid p(s) = 0 \text{ for all } s \in \Gamma\}.$$

Similarly, we let $\Sigma_{n,2d}(\Gamma)$ be the exposed face of $\Sigma_{n,2d}$ consisting of forms that vanish at all points of Γ :

$$\Sigma_{n,2d}(\Gamma) = \{p \in \Sigma_{n,2d} \mid p(s) = 0 \text{ for all } s \in \Gamma\}.$$

Moreover, any exposed face of $P_{n,2d}$ has a description of the above form, and the set Γ can be chosen to be finite [6, Chapter 4]. We note that, despite this simple description of exposed faces, the full facial structure of $P_{n,2d}$ “should” be very difficult to fully describe since – as already mentioned – the problem of testing for nonnegativity is known to be NP-hard. Furthermore, even for the exposed faces $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$, except for the simple cases of $n = 2$ and $2d = 2$, the possible dimensions of these faces have not been investigated yet. We close this gap by deriving estimates for the dimensions of the faces $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ and by establishing dimensional differences between those faces in many cases. It is worth remarking that also Hilbert’s original proof [11] of existence of nonnegative polynomials that are not sums of squares can be viewed as establishing a dimensional gap of this type. This dimensional point of view was first made explicit in [19].

For a generic set Γ we reduce the question of dimensions of $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ to the question of dimensions of the degree $2d$ components of certain ideals associated with Γ . To simplify notation, we set $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$. For an ideal $I \subset \mathbb{R}[x]$, let I^2 denote the second ordinary power of I , and let $I^{(2)}$ denote the second symbolic power of I . Moreover, let I_d denote the homogeneous degree d part of I .

If $I(\Gamma) \subset \mathbb{R}[x]$ is the vanishing ideal of a finite set of distinct points $\Gamma \subset \mathbb{RP}^{n-1}$, then the second symbolic power $I^{(2)}(\Gamma)$ of $I(\Gamma)$ is the ideal of all forms in $\mathbb{R}[x]$ vanishing at every point of Γ to order at least 2:

$$I^{(2)}(\Gamma) = \{p \in \mathbb{R}[x] \mid \nabla p(s) = 0 \text{ for all } s \in \Gamma\}.$$

Since every nonnegative form that is zero on $s \in \Gamma$ must vanish to order 2 on s , it follows that the face $P_{n,2d}(\Gamma)$ is contained in the degree $2d$ part of $I^{(2)}(\Gamma)$:

$$P_{n,2d}(\Gamma) \subset I_{2d}^{(2)}(\Gamma).$$

On the other hand, we know that the face $\Sigma_{n,2d}(\Gamma)$ is contained in the $2d$ part of the ordinary square of $I(\Gamma)$, i.e.,

$$\Sigma_{n,2d}(\Gamma) \subset I_{2d}^2(\Gamma).$$

It is easy to see that this inclusion is actually full-dimensional, since we can choose a basis of $I_{2d}^2(\Gamma)$ consisting of squares and any nonnegative linear combination of these squares lies in $\Sigma_{n,2d}(\Gamma)$. More precisely, the following is true.

Proposition 1.1. *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set. Then $\Sigma_{n,2d}(\Gamma)$ is a full-dimensional convex cone in the vector space of all forms of degree $2d$ in $I_{2d}^2(\Gamma)$:*

$$\dim \Sigma_{n,2d}(\Gamma) = \dim I_{2d}^2(\Gamma).$$

As for sums of squares, we pose the question, under which assumptions $P_{n,2d}(\Gamma)$ is a full-dimensional subcone of $I_{2d}^{(2)}(\Gamma)$. In order to answer this question, the following crucial definition is required.

Definition 1.2. Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set of distinct points and $I = I(\Gamma) \subset \mathbb{R}[x]$ be the vanishing ideal of Γ . We call $\Gamma \subset \mathbb{RP}^{n-1}$ *d-independent* if Γ satisfies the following two conditions:

- (1.1) The forms in I_d share no common zeroes in \mathbb{CP}^{n-1} outside of Γ . In other words, the conditions of vanishing at Γ force no additional zeroes on forms of degree d in $H_{n,d}$.
- (1.2) For any $s \in \Gamma$ the forms that vanish to order 2 on s and vanish at the rest of Γ to order 1 form a vector space of codimension $|\Gamma| + n - 1$ in $H_{n,d}$.

The second condition in the above definition simply states that the constraints of vanishing at Γ and additionally double vanishing at any point $s \in \Gamma$ are all linearly independent.

The next proposition provides a sufficient condition for full-dimensionality of a face $P_{n,2d}(\Gamma)$ in $I_{2d}^{(2)}(\Gamma)$.

Proposition 1.3. Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a *d-independent* set. Then $P_{n,2d}(\Gamma)$ is a full-dimensional convex cone in $I_{2d}^{(2)}(\Gamma)$:

$$\dim P_{n,2d}(\Gamma) = \dim I_{2d}^{(2)}(\Gamma).$$

Though we have seen that for nonnegative forms the condition for full-dimensionality of $P_{n,2d}(\Gamma)$ in $I_{2d}^{(2)}(\Gamma)$ is not as simple as for full-dimensionality of $\Sigma_{n,2d}(\Gamma)$ in $I_{2d}^2(\Gamma)$, it follows from the next two results that for generic Γ the dimensions of $P_{n,2d}(\Gamma)$ and $I_{2d}^{(2)}(\Gamma)$ agree.

Corollary 1.4. The set of *d-independent* configurations of k points in \mathbb{RP}^{n-1} is a Zariski open subset of $(\mathbb{RP}^{n-1})^k$.

Proposition 1.5. Let Γ be a generic collection of points in \mathbb{RP}^{n-1} such that $|\Gamma| \leq \binom{n+d-1}{d} - n$. Then Γ is *d-independent*.

In view of Propositions 1.1, 1.3 and 1.5 our original question of finding a dimensional difference between the faces $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ can be reduced to the following:

Question 1.1. Let $\Gamma \subset \mathbb{RP}^{n-1}$ (or equivalently \mathbb{CP}^{n-1}) be a generic set of points such that $|\Gamma| \leq \binom{n+d-1}{d} - n$, and let $I(\Gamma)$ be the vanishing ideal of Γ . For what values of $|\Gamma|$ do we have

$$I_{2d}^{(2)}(\Gamma) = I_{2d}^2(\Gamma)?$$

Indeed, for $n = 3$, we show that this is true for all *d-independent* sets up to a certain size.

Theorem 1.6. Let Γ be a *d-independent* set of points in \mathbb{RP}^2 such that $|\Gamma| \leq \binom{d+1}{2}$. Then

$$\dim I_{2d}^{(2)}(\Gamma) = \dim I_{2d}^2(\Gamma).$$

Moreover, if $\binom{d+1}{2} + 1 \leq |\Gamma| \leq \binom{d+1}{2} + (d-2)$, then $\dim I_{2d}^{(2)}(\Gamma) > \dim I_{2d}^2(\Gamma)$.

As already explained, the next corollary is an immediate consequence of Theorem 1.6.

Corollary 1.7. Let $\Gamma \subset \mathbb{RP}^2$ be *d-independent* with $|\Gamma| \leq \binom{d+1}{2}$. Then

$$\dim P_{3,2d}(\Gamma) = \dim \Sigma_{3,2d}(\Gamma).$$

Furthermore, for $\binom{d+1}{2} + 1 \leq |\Gamma| \leq \binom{d+1}{2} + (d-2)$ we have

$$\dim P_{3,2d}(\Gamma) > \dim \Sigma_{3,2d}(\Gamma).$$

Similarly, for $n = 4$, we derive the minimal size of a d -independent set Γ such that the dimensions of the ideals $I_4^2(\Gamma)$ and $I_4^{(2)}(\Gamma)$ are distinct. We will prove that if $\Gamma \subset \mathbb{RP}^3$ with $|\Gamma| \leq 6$ is in general linear position, then it is d -independent.

Theorem 1.8. *Let $\Gamma \subset \mathbb{RP}^3$ be a finite set in general linear position. Then the following hold:*

(i) *If $|\Gamma| = 6$, then*

$$\dim I_4^2(\Gamma) = 10 < 11 = \dim I_4^{(2)}(\Gamma).$$

(ii) *If $|\Gamma| \leq 5$, then*

$$\dim I_4^2(\Gamma) = \dim I_4^{(2)}(\Gamma).$$

As a direct consequence, we obtain the following corollary.

Corollary 1.9. *Let $\Gamma \subset \mathbb{RP}^3$ be a finite set in general linear position. Then the following hold:*

(i) *If $|\Gamma| = 6$, then*

$$\dim \Sigma_{4,4}(\Gamma) = 10 < 11 = \dim P_{4,4}(\Gamma).$$

(ii) *If $|\Gamma| \leq 5$, then*

$$\dim \Sigma_{4,4}(\Gamma) = \dim P_{4,4}(\Gamma).$$

The paper is structured as follows. Section 2 focuses on proving that d -independence is a sufficient condition for full-dimensionality of $P_{n,2d}(\Gamma)$ in $I_{2d}^{(2)}(\Gamma)$. Additionally, properties of d -independent sets are studied, in particular, for sets with bounded size, d -independence turns out to be a generic condition. In Section 3, ternary forms are considered. Among other results, we prove Theorem 1.6, which completely characterizes the occurrence of dimensional differences between exposed faces $P_{3,2d}(\Gamma)$ and $\Sigma_{3,2d}(\Gamma)$. In Section 4, we are interested in the case $(n, 2d) = (4, 4)$ and derive analogous statements as for ternary forms (Theorem 1.8), which yield a complete classification also in this case. In the two smallest cases $((n, 2d) \in \{(3, 6), (4, 4)\})$, we show that the maximal dimensional difference between the exposed faces is exactly one. Section 5 provides explicit characterizations of these one-dimensional differences, yielding nonnegative polynomials that are not sums of squares. Finally, in Section 6, we compute some a priori bounds for the possible dimensional differences as well as naive bounds for the minimum cardinality of a d -independent set such that a dimensional difference occurs. Moreover, we state some open questions for further research.

2. DIMENSION OF FACES OF $P_{n,2d}$

In this section, we study the inclusion $P_{n,2d}(\Gamma) \subset I_{2d}^{(2)}(\Gamma)$ in more detail. In particular, we are interested in the question of when this inclusion is full-dimensional. Moreover, the proofs of Propositions 1.3 and 1.5, such as Corollary 1.4 are provided.

2.1. Sum of Squares Certificate. In this section, we provide the proof of Proposition 1.3.

Let Γ be a finite set of points in \mathbb{RP}^{n-1} and consider $I_{2d}^{(2)}(\Gamma)$, the vector space of forms of degree $2d$ vanishing at Γ with multiplicity at least 2. Every double zero forces n linear conditions on forms vanishing at Γ . Since not all of these conditions are necessarily independent, the following inequality always holds:

$$(2.1) \quad \dim I_{2d}^{(2)}(\Gamma) \geq \dim H_{n,2d} - n|\Gamma|.$$

However, generically – with a small list of exceptions – the Alexander-Hirschowitz Theorem [15] tells us that equality holds in (2.1).

We establish full-dimensionality of $P_{n,2d}(\Gamma)$ in $I_{2d}^{(2)}(\Gamma)$ by finding a form $p \in P_{n,2d}(\Gamma)$ to which we can add a suitably small multiple of any double vanishing form such that it will remain nonnegative:

$$p + \epsilon q \in P_{n,2d}(\Gamma) \text{ for some sufficiently small } \epsilon \text{ and any } q \in I_{2d}^{(2)}(\Gamma).$$

The form p can be viewed as a certificate of full-dimensionality of $P_{n,2d}(\Gamma)$ in $I_{2d}^{(2)}(\Gamma)$. The important point is that p can be any form, in particular, we will focus on finding such p that is a sum of squares. This approach follows that of [19] and, indeed, it can be traced to the original proof of Hilbert [11].

For a form p , let the Hessian H_p of p be the matrix of second derivatives of p :

$$H_p = (h_{ij}), \quad \text{where} \quad h_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j}.$$

We note that if a form p vanishes at a point s , then, by homogeneity, p needs to vanish at a line through s . Therefore, s lies in the kernel of the Hessian of p at s : $H_p(s)s = 0$.

If a form p is nonnegative, then its Hessian at any zero s is positive semidefinite since 0 is a minimum for p . We call a nonnegative form p *round* at a zero $s \in \mathbb{RP}^{n-1}$ if the Hessian of p at s is positive definite on the subspace s^\perp of vectors perpendicular to s , i. e.,

$$p \text{ is round at a zero } s \Leftrightarrow y^T H_p(s) y > 0 \text{ for all } y \in s^\perp.$$

For a form p , we let $Z(p)$ denote the real projective variety of p .

We will need the following “extension lemma”, which follows from Lemma 3.1 of [19].

Lemma 2.1. *Let $p \in P_{n,2d}$ be a nonnegative form with a finite zero set $Z(p)$ and suppose that p is round at every point in $Z(p)$. Furthermore, let q be a form such that q vanishes to order 2 on $Z(p)$. Then, for a sufficiently small ϵ , the form $p + \epsilon q$ is nonnegative.*

From this, we infer the following immediate corollary, which will be crucial for the proof of Proposition 1.3.

Corollary 2.2. *Let Γ be a finite set in \mathbb{RP}^{n-1} . Suppose that there exists a nonnegative form p in $P_{n,2d}(\Gamma)$ such that $Z(p) = \Gamma$ and p is round at every point $s \in \Gamma$. Then the face $P_{n,2d}(\Gamma)$ is full-dimensional in the vector space $I_{2d}^{(2)}(\Gamma)$.*

Proof. Let $p \in P_{n,2d}(\Gamma)$ be as in the assumptions. Then, by Lemma 2.1, for any $q \in I_{2d}^{(2)}(\Gamma)$ we have $p + \epsilon q \in P_{n,2d}(\Gamma)$ for sufficiently small ϵ . Since $P_{n,2d}(\Gamma)$ is a convex set, it follows that it is full-dimensional in $I_{2d}^{(2)}(\Gamma)$. \square

We can finally provide the proof of Proposition 1.3.

Proof of Proposition 1.3. Let q_1, \dots, q_k be a basis of $I_d(\Gamma)$. We claim that $p = \sum_{i=1}^k q_i^2$ has the properties of Corollary 2.2 and therefore, the convex cone $P_{n,2d}(\Gamma)$ is full-dimensional in $I_{2d}^{(2)}(\Gamma)$.

Since Γ forces no additional zeroes and since q_1, \dots, q_k is a basis of $I_d(\Gamma)$, it follows that the forms q_i have no common zeroes outside of Γ and thus $Z(p) = \Gamma$.

Now choose $s \in \Gamma$. It remains to show that p is round at s , i. e., $H_p(s)$ is positive definite on s^\perp . Since the forms in $I_d(\Gamma)$ that double vanish at s form a vector space of codimension $n - 1$ in $I_d(\Gamma)$, we see that for $1 \leq i \leq k$ the gradients of q_i at s must span a vector space of dimension $n - 1$. Since, by Euler’s identity (see, e. g., [10, Lemma 11.4]), $\langle \nabla q_i, s \rangle = 0$ for all i , this implies that the gradients actually span s^\perp .

Note that the Hessian of p is the sum of the Hessians of q_i^2 , i. e.,

$$H_p = \sum_{i=1}^k H_{q_i^2}.$$

Since $q_i(s) = 0$ for all i and $s \in \Gamma$, we conclude that

$$\frac{\partial^2 q_i^2}{\partial x_l \partial x_j}(s) = 2 \frac{\partial q_i}{\partial x_l}(s) \frac{\partial q_i}{\partial x_j}(s).$$

Therefore, we see that the Hessian of q_i^2 at any $s \in \Gamma$ is actually double the tensor of the gradient of q_i at s with itself:

$$H_{q_i^2}(s) = 2 \nabla q_i \otimes \nabla q_i(s).$$

It is now straightforward to verify that

$$\nabla q_j(s)^T H_{q_i^2}(s) \nabla q_j(s) > 0$$

for all $1 \leq i, j \leq k$ and $s \in \Gamma$, which shows the claim. \square

From now on, we will focus on the study of the degree $2d$ part of the second symbolic power $I_{2d}^{(2)}(\Gamma)$ since, by Proposition 1.3, we have the equality

$$\dim P_{n,2d}(\Gamma) = \dim I_{2d}^{(2)}(\Gamma),$$

whenever Γ is a finite d -independent set. We first provide the following equivalent characterization of d -independence.

Proposition 2.3. *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set of points and let $J \subset \mathbb{R}[x]$ be the ideal generated by $I_d(\Gamma)$. Then Γ is d -independent if and only if the Hilbert polynomial of $\mathbb{R}[x]/J$ is equal to $|\Gamma|$.*

Proof. Let J be the ideal generated by $I_d(\Gamma)$. Let $I_{\mathbb{C},d}(\Gamma)$ be the set of all degree d complex forms vanishing at Γ . We first observe that linear combinations of forms in $I_d(\Gamma)$ taken with complex coefficients generate $I_{\mathbb{C},d}(\Gamma)$. Therefore, if we let $J_{\mathbb{C}}$ be the complex ideal generated $I_{\mathbb{C},d}(\Gamma)$, then it suffices to show that the Hilbert polynomial of $\mathbb{C}[x]/J_{\mathbb{C}}$ is $|\Gamma|$.

We now apply Bertini's Theorem (see, e.g., [9, Theorem 17.16]) to the linear system of divisors $I_{\mathbb{C},d}(\Gamma)$. Since Γ is d -independent, it follows that a general element of $I_{\mathbb{C},d}(\Gamma)$ is non-singular at every point of Γ and thus we may find a smooth hypersurface $f_1 \in I_{\mathbb{C},d}(\Gamma)$. Now d -independence guarantees that a general form in $I_{\mathbb{C},d}(\Gamma)$ intersects f_1 smoothly and therefore we apply Bertini's Theorem again to find $f_2 \in I_{\mathbb{C},d}(\Gamma)$ such that $f_1 \cap f_2$ is smooth. We proceed in this way repeatedly applying Bertini's Theorem until we end up with a transverse zero-dimensional intersection $V = f_1 \cap \dots \cap f_{n-1}$. By construction we have $\Gamma \subseteq V$.

Since $J_{\mathbb{C}}$ defines a zero-dimensional ideal and all forms in $J_{\mathbb{C}}$ vanish at Γ , it follows that the Hilbert polynomial of $\mathbb{C}[x]/J_{\mathbb{C}}$ is a constant and it is greater or equal than $|\Gamma|$. Since we have $\langle f_1, \dots, f_{n-1} \rangle \subset J_{\mathbb{C}}$ and the ideal generated by f_1, \dots, f_{n-1} is radical, it follows that we just need to show that for all $s \in V \setminus \Gamma$ there exists $g \in J_{\mathbb{C}}$ such that $g(s) \neq 0$ and $g(z) = 0$ for all $z \in V \setminus \{s\}$. Since Γ is d -independent, it suffices to show for all $s \in V \setminus \Gamma$ there exists $h \in I_d(\Gamma)$ such that $h(s) \neq 0$. Also there exists $h' \in \mathbb{C}[x_1, \dots, x_n]$ such that $h'(s) = 1$ and $h'(z) = 0$ for all $z \in V \setminus \{s\}$. Therefore $hh' \in J_{\mathbb{C}}$ and one direction of the proposition follows.

Now suppose that Γ is not d -independent. First, if all forms in $I_d(\Gamma)$ vanish at a point not in Γ , then all forms in J vanish on at least $|\Gamma| + 1$ points and therefore the Hilbert function of $\mathbb{R}[x]/J$ is at least $|\Gamma| + 1$ for all large enough degrees. Therefore the Hilbert polynomial of $\mathbb{R}[x]/J$ is not $|\Gamma|$. Now suppose that for some $s \in \Gamma$ and some nonzero $w \in \mathbb{R}^n$ we have $\langle \nabla p(s), w \rangle = 0$ for all $p \in I_d(\Gamma)$. Then again we find that the Hilbert function of $\mathbb{R}[x]/J$ is at least $|\Gamma| + 1$ for all high enough degrees. \square

We remark that in the above proof h' may be chosen such that $\deg h' \leq (n-1)(d-1)$. Therefore we have $\deg hh' \leq (n-1)(d-1) + d$ and for all degrees $k \geq (n-1)(d-1) + d$ the Hilbert function of $\mathbb{C}[x]/J_{\mathbb{C}}$ (and of $\mathbb{R}[x]/J$) evaluated at k must be equal to $|\Gamma|$. Thus using standard methods (vanishing determinants) we can express the set of all configurations of k points in \mathbb{RP}^{n-1} that are

d -independent as a complement of a closed algebraic set. Hence, the set of configurations Γ of k points in \mathbb{RP}^{n-1} that are d -independent is Zariski open, which proves Corollary 1.4.

2.2. A d -independent set of size $\binom{n+d-1}{d} - n$. The aim of this section is to provide the proof of Proposition 1.5. For this goal, we will construct an example of a d -independent set of cardinality $\binom{n+d-1}{d} - n$. Since, by Corollary 1.4, being d -independent is a Zariski open condition, this will already show the claim.

Define $\bar{S}_{n,d}$ to be the set of points in \mathbb{RP}^{n-1} that correspond to nonnegative integer partitions of d :

$$\bar{S}_{n,d} = \left\{ [\alpha_1 : \dots : \alpha_n] \in \mathbb{RP}^{n-1} \mid \alpha_i \in \mathbb{Z}, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = d \right\}.$$

We can think of the points in $\bar{S}_{n,d}$ as all the possible exponent choices for monomials in n variables of degree d . Therefore, $\bar{S}_{n,d}$ contains $\binom{n+d-1}{d}$ points.

Now let $S_{n,d}$ be the set of points in \mathbb{RP}^{n-1} that correspond to partitions of d with at least 2 nonzero parts. The points in $S_{n,d}$ again correspond to monomials of degree d but we exclude the monomials of the form x_i^d . Therefore, $S_{n,d}$ contains $\binom{n+d-1}{d} - n$ points.

Proposition 2.4. *The set $S_{n,d}$ is d -independent.*

The proof of the above proposition requires some additional results. The following proposition is taken from [17, p. 31] and has been known for at least a hundred years. We reproduce the proof below.

Proposition 2.5. *There are no nontrivial forms in $H_{n,d}$ that vanish at $\bar{S}_{n,d}$. In other words, $I_d(\bar{S}_{n,d}) = 0$.*

Proof. For every point $s = [s_1 : \dots : s_n] \in \bar{S}_{n,d}$, we will construct a form $p_s \in \mathbb{K}[x]_{n,d}$ that vanishes at all points in $\bar{S}_{n,d}$ except for s . This shows that the conditions of vanishing at any point in $\bar{S}_{n,d}$ are linearly independent and since $|\bar{S}_{n,d}| = \dim H_{n,d}$, we see that $\dim I_d(\bar{S}_{n,d}) = 0$.

Let $M = x_1 + \dots + x_n$. For $i = 1, \dots, n$, let h_i be the form defined as follows:

$$h_i = \prod_{k=0}^{s_i-1} (dx_i - kM).$$

It is clear that the degree of h_i is s_i and h_i vanishes at all partitions in $\bar{S}_{n,d}$ with i -th part less than s_i . Now let p_s be defined as:

$$p_s = \prod_{i=1}^n h_i.$$

The form p_s has degree $\sum_{i=1}^n s_i = d$, and it does not vanish at s . However, for any other partition of d , there exists i such that the i -th part is less than s_i . Then, h_i will vanish for that i and, thus, p_s will vanish at any partition of d except for s . \square

As in the proof of Proposition 2.5 let $M = x_1 + \dots + x_n$. For $i = 1, \dots, n$, define a form Q_i as follows:

$$(2.2) \quad Q_i = \prod_{k=0}^{d-1} (dx_i - kM).$$

We observe that each Q_i vanishes at $S_{n,d}$. Indeed, let $s = [s_1 : \dots : s_n] \in S_{n,d}$ and consider $Q_i(s)$. We know that $M(s) = d$, because points in $S_{n,d}$ are partitions of d , and, therefore, the factor of Q_i that corresponds to $k = s_i$ will vanish at s , forcing $Q_i(s) = 0$. Hence, $Q_i \in I_d(S_{n,d})$ for all $i = 1, \dots, n$.

We will now show that the forms Q_i actually form a basis of $I_d(S_{n,d})$. The fact that we have such a nicely factoring basis is what, eventually, allows us to prove that $S_{n,d}$ is d -independent.

Proposition 2.6. *The forms Q_i form a basis of $I_d(S_{n,d})$.*

Proof. We first show that the forms Q_i are linearly independent. Let e_1, \dots, e_n be the standard basis vectors of \mathbb{RP}^{n-1} . It is easy to see that $Q_i(e_j) = 0$ for $i \neq j$ since x_i divides Q_i . On the other hand, $Q_i(e_i) = d!$. Therefore, if there exist $\alpha_i \in \mathbb{R}$ such that $\alpha_1 Q_1 + \dots + \alpha_n Q_n = 0$, then, by evaluating this linear combination at e_i , we see that $\alpha_i = 0$ and this works for all i . Thus, the forms Q_i are linearly independent.

We now show that the forms Q_i span $I_d(S_{n,d})$. Let $p \in I_d(S_{n,d})$ and let $\beta_i = p(e_i)$. Consider the form

$$\bar{p} = p - \sum_{i=1}^n \frac{\beta_i}{d!} Q_i.$$

It is clear that \bar{p} vanishes at the standard basis vectors e_i . Therefore, \bar{p} vanishes not only on $S_{n,d}$ but also on $\bar{S}_{n,d}$. By Proposition 2.5, it follows that $\bar{p} = 0$ and, therefore, p is in the span of Q_i . \square

We now show that the set $S_{n,d}$ satisfies the two conditions of d -independence from Definition 1.2.

Lemma 2.7. *The set $S_{n,d}$ forces no additional zeroes for forms of degree d .*

Proof. Since by Proposition 2.6 the forms Q_i form a basis of $I_d(S_{n,d})$, the statement of the lemma is equivalent to showing that $S_{n,d}$ is projectively equal to $\cap_{i=1}^n Z(Q_i)$.

Let $v = [v_1 : \dots : v_n] \in \cap_{i=1}^n Z(Q_i)$ be a nonzero point and first suppose that $v_1 + \dots + v_n = 0$, i. e., $M(v) = 0$. Therefore, by Equation (2.2), we see that $Q_i(v) = d^d v_i^d$. Since by assumption $Q_i(v) = 0$ for all i , it follows that $v = 0$, which is a contradiction.

Now suppose that $v_1 + \dots + v_n \neq 0$. By homogeneity we can assume that $v_1 + \dots + v_n = d$. In this case, from Equation (2.2), it follows that $Q_i(v) = d^d v_i(v_i - 1) \dots (v_i - d + 1)$. Since $Q_i(v) = 0$ for all i , we infer that each v_i is a nonnegative integer between 0 and $d - 1$ and $v_1 + \dots + v_n = d$. In other words, $v \in S_{n,d}$. \square

We know that $|S_{n,d}| = \binom{n+d-1}{d} - n$. For the second condition of d -independence, we need to show that for any $s \in S_{n,d}$ the vector space of forms double vanishing at s and vanishing at the rest of $S_{n,d}$ with multiplicity 1 has codimension $|S_{n,d}| + n - 1 = \binom{n+d-1}{d} - 1$ in $H_{n,d}$. Since $\dim H_{n,d} = \binom{n+d-1}{d}$, we thus need to show that the vector space of forms double vanishing at any $s \in S_{n,d}$ and vanishing at the rest of $S_{n,d}$ with multiplicity 1 is one-dimensional. This will follow from the next lemma.

Lemma 2.8. *For every point $s \in S_{n,d}$, there is a unique (up to a constant multiple) form in $I_d(S_{n,d})$ singular at s .*

Proof. Let $s = [s_1 : \dots : s_n] \in S_{n,d}$ and let $p \in I_d(S_{n,d})$ be a form singular at s .

Since by Proposition 2.6 the forms Q_i form a basis of $I_d(S_{n,d})$, we may assume that $p = \alpha_1 Q_1 + \dots + \alpha_n Q_n$ for certain $\alpha_i \in \mathbb{R}$. Now let $A = (a_{ij})$ be the $n \times n$ matrix with entries

$$a_{ij} = \frac{\partial Q_i}{\partial x_j}(s).$$

The statement of the lemma is equivalent to showing that $\text{rank } A = n - 1$. Recall from Equation (2.2) the definition of Q_i :

$$Q_i = \prod_{k=0}^{d-1} dx_i - kM.$$

The form Q_i vanishes at s , because the term $dx_i - s_i M$ corresponding to $k = s_i$ vanishes at s . Therefore, the only nonzero term in $\frac{\partial Q_i}{\partial x_j}$ evaluated at s will come from differentiating out $dx_i - s_i M$. Now, for $1 \leq i \leq n$, let

$$P_{s_i} = \frac{Q_i}{dx_i - s_i M}.$$

We observe that $P_{s_i}(s) \neq 0$ since we removed from Q_i the only factor that vanishes at s .

Recall that $M = x_1 + \dots + x_n$ and, therefore, if we differentiate out $dx_i - s_i M$ from Q_i with respect to x_j and evaluate it at s , we see that

$$\frac{\partial Q_i}{\partial x_j}(s) = \begin{cases} P_{s_i}(s)(d - s_j) & \text{if } i = j \\ -P_{s_i}(s)s_j & \text{if } i \neq j. \end{cases}$$

Since $P_{s_i}(s) \neq 0$, we can divide the i -th row of A by $P_{s_i}(s)$ to obtain a matrix $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} d - s_j & \text{if } i = j \\ -s_j & \text{if } i \neq j. \end{cases}$$

By construction, $\text{rank } B = \text{rank } A$.

Since s is a partition of d , it is clear that the vector consisting of all 1s is in the kernel of B . Now let $C = (c_{ij})$ be the matrix with j -th column having the same entry s_j , i.e., $c_{ij} = s_j$. We observe that the rank of C is 1 and $B = dI - C$, where I is the identity matrix. Therefore, we know that $\text{rank } B \geq \text{rank } I - \text{rank } C = n - 1$. Since we already found a vector in the kernel of B , it follows that the rank of B is $n - 1$. \square

We have now shown that the set $S_{n,d}$ is d -independent and together with Corollary 1.4 this shows that d -independence is a generic condition for sets of k points in \mathbb{RP}^{n-1} with $k \leq \binom{n+d-1}{d} - n$. In particular, this proves Proposition 1.5.

3. COMPLETE CHARACTERIZATION OF TERNARY FORMS

In this section, we study the case of ternary forms and provide a complete characterization for the occurrence of dimensional differences between exposed faces of the cones $P_{3,2d}$ and $\Sigma_{3,2d}$. The desired characterization will require some preparatory lemmas.

Lemma 3.1. *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set and $I = I(\Gamma) \subset \mathbb{R}[x]$ be the vanishing ideal. Let $\overline{I^m}$ be the saturation of the ordinary m -th power of I . Then*

$$\overline{I^m} = I^{(m)}.$$

Proof. Consider the primary decomposition $I = \bigcap_{p \in S} I(p)$, where $I(p) = \{f \in \mathbb{R}[x] \mid f(p) = 0\}$. Then, by definition of symbolic powers, we have $I^{(m)} = \bigcap_{p \in S} I(p)^m$. Moreover, the primary decomposition of I^m contains $I(p)^m$ for all $p \in S$ and an additional component Q associated to

the maximal homogeneous ideal $\mathfrak{m} = (x_1, \dots, x_n)$. By definition $\overline{I^m} = \bigcup_{j \geq 0} I^m : \mathfrak{m}^j$. From the previous discussion, we obtain

$$\begin{aligned} I^m : \mathfrak{m}^j &= \left(\bigcap_{p \in S} I(p)^m \cap Q \right) : \mathfrak{m}^j \\ &= \left(\bigcap_{p \in S} I(p)^m : \mathfrak{m}^j \right) \cap Q : \mathfrak{m}^j \\ &= (I^{(m)} : \mathfrak{m}^j) \cap (Q : \mathfrak{m}^j) \end{aligned}$$

Since Q is \mathfrak{m} -primary, for sufficiently large j , it follows that $Q : \mathfrak{m}^j = \mathbb{R}[x]$. Moreover, $I^{(m)} : \mathfrak{m}^j = I^{(m)}$, thus, $I^m : \mathfrak{m}^j = I^{(m)}$, for sufficiently large j . \square

Note that, as a consequence of the above lemma, we clearly have that $(\overline{I^2})_{2d} = (I^{(2)})_{2d}$.

Given an ideal $I \subset \mathbb{R}[x]$, let $\alpha(I)$ be the minimum degree of a generator of I . We make the following simple observation.

Lemma 3.2. *Let $I \subset \mathbb{R}[x]$ be a homogeneous ideal and let $m = \alpha(I)$. Then $(I^2)_{2m} = (I_m)^2$.*

Proof. Since $I_m \subset I$, we always have the inclusion $(I_m)^2 \subset (I^2)_{2m}$. On the other hand, if $f \in (I^2)_{2m}$, then we can write $f = \sum_{a \in J} g_a \cdot h_a$ for certain polynomials $g_a, h_a \in I$ and $J \subset \mathbb{N}^n$ finite. Moreover, since $2m = \deg(f) = \deg(g_a) + \deg(h_a)$ and $\alpha(I) = m$, we conclude $\deg(f_a) = \deg(g_a) = m$ for all $a \in J$. Hence, $g_a, h_a \in I_m$ for all $a \in J$. \square

Recall that for an ideal $I \subset \mathbb{R}[x]$ the *saturation degree* of I is defined as

$$\text{satdeg}(I) = \min\{s \mid \overline{I_s} = I_s\},$$

see, e.g., [8]. Moreover, we use $\text{reg}(I)$ to denote the (Castelnuovo-Mumford) regularity of I .

Lemma 3.3. *Let $\Gamma \subset \mathbb{RP}^{n-1}$ be a finite set and let $I = I(\Gamma) \subset \mathbb{R}[x]$ be the vanishing ideal. If $\alpha(I) = \text{reg}(I)$, then $\text{satdeg}(I^2) \leq \alpha(I^2) = 2\alpha(I)$.*

Proof. As we have seen in the proof of Lemma 3.1, it holds that $I^2 = I^{(2)} \cap Q$, where Q is \mathfrak{m} -primary. More precisely, one has $Q = \mathfrak{m}^{2\alpha(I)}$. So, for $j \in \mathbb{N}$, one has $(I^2)_j = (I^{(2)} \cap \mathfrak{m}^{2\alpha(I)})_j$. For $j \geq 2\alpha(I)$, the ideal $\mathfrak{m}^{2\alpha(I)}$ contains all monomials of degree j . Therefore, $(I^2)_j = I_j^{(2)}$ in this case and, by Lemma 3.1, it follows that $(I^2)_j = (\overline{I^2})_j$, i.e., $\text{satdeg}(I^2) \leq 2\alpha(I)$. The equality $\alpha(I^2) = 2\alpha(I)$ is true in general. \square

We will need the following definition of a set $\Gamma \subset \mathbb{R}^n$ being in general linear position, which naturally extends to \mathbb{RP}^{n-1} .

Definition 3.4. Let $\Gamma \subset \mathbb{R}^n$ be a finite set. Γ is in *general linear position* if one of the following conditions holds

- (i) $|\Gamma| \leq n$ and $\dim \text{Aff}(\Gamma) = |\Gamma| - 1$
- (ii) $|\Gamma| \geq n + 1$ and no n of the points in Γ lie on a common hyperplane.

Note that any set Γ with $|\Gamma| \leq n$ that is in general linear position can be extended to a set in general linear position of greater cardinality.

Lemma 3.5. *Let $\Gamma \subset \mathbb{RP}^2$ be in general linear position with $|\Gamma| = \binom{d+1}{2}$. Then $\text{reg}(I_d(\Gamma)) = d$.*

Proof. Due to the minimal resolution conjecture for \mathbb{RP}^2 (see, e.g., [1, 20]), for $\Gamma \subset \mathbb{RP}^2$ it holds that $\text{reg}(I_d(\Gamma)) \in \{d, d+1\}$. For $|\Gamma| = \binom{d+1}{2}$ it is shown in [13, Section 3] that, indeed, $\text{reg}(I_d(\Gamma)) = d$. \square

We can now prove Theorem 1.6.

Proof of Theorem 1.6. Let $\Gamma \subset \mathbb{RP}^2$ be such that $|\Gamma| = \binom{d+1}{2} - k$ for $0 \leq k \leq \binom{d+1}{2} - 1$. We first prove that $\dim I_{2d}^2(\Gamma) = \dim I_{2d}^{(2)}(\Gamma)$. We show the claim by induction on k . First assume $k = 0$, i. e., $|\Gamma| = \binom{d+1}{2}$. By Lemma 3.5, we have $\alpha(I_d(\Gamma)) = \text{reg}(I_d(\Gamma)) = d$ and, from Lemmas 3.2 and 3.3, we infer that $I_{2d}^{(2)}(\Gamma) = I_{2d}^2(\Gamma)$.

Now suppose that the claim holds for a fixed k and hence we have $\dim I_{2d}^2(\Gamma) = \dim I_{2d}^{(2)}(\Gamma) = \binom{d+2}{2} + 3k$ for any d -independent set $\Gamma \subset \mathbb{RP}^2$ with $|\Gamma| = \binom{d+1}{2} - k$. For the induction step $k \mapsto k+1$, let $\Gamma \subset \mathbb{RP}^2$ be d -independent with $|\Gamma| = \binom{d+1}{2} - (k+1)$. Let $p \in \mathbb{RP}^2$ such that the set $T = \Gamma \cup \{p\}$ is d -independent and in general linear position. Note that $|T| = \binom{d+1}{2} - k$. Since Γ and T are d -independent, we have

$$\dim I_d(T) = d + 1 + k \text{ and } \dim I_d(\Gamma) = d + 2 + k.$$

Furthermore, we know that $\dim I_{2d}^{(2)}(T) = \binom{d+2}{2} + 3k$ by hypothesis. Now, let $Q_1 \in I_d(T)$ with the additional property that one of its partial derivatives vanishes at p . Without loss of generality, assume $\frac{\partial Q_1}{\partial x_1}(p) = 0$. We can extend Q_1 to a basis $B = \{Q_1, Q_2, \dots, Q_{d+1+k}\}$ of $I_d(T)$. Note that there must be at least one basis element Q_j , $j \neq 1$, such that $\frac{\partial Q_j}{\partial x_1}(p) \neq 0$. Otherwise, we arrive at a contradiction in regard to d -independence of T . By assumption, there exist $\binom{d+2}{2} + 3k$ pairwise products of elements of B forming a basis of $I_{2d}^2(T)$. Let \tilde{B} denote this basis of $I_{2d}^2(T)$. Furthermore, extend B to a basis of $I_d(\Gamma)$ by adding a suitable form $Q \in H_{3,d}$. Observe that $Q(p) \neq 0$ since Γ is d -independent. We claim that there exist two forms $Q_l, Q_m \in B$ such that the set

$$L = \tilde{B} \cup \{Q_l Q, Q_m Q, Q^2\}$$

forms a basis of $I_{2d}^2(\Gamma)$. Suppose that this set L is linearly dependent for any choice of $Q_l, Q_m \in B$. Hence, we have

$$\sum_{Q_i Q_j \in \tilde{B}} \alpha_{ij} Q_i Q_j + \alpha_{l,m}^{(l)} Q_l Q + \alpha_{l,m}^{(m)} Q_m Q + \alpha Q^2 = 0$$

for a nontrivial set of coefficients $(\alpha_{ij}, \alpha_{l,m}^{(l)}, \alpha_{l,m}^{(m)}, \alpha)$. To simplify notation we use $P_{l,m}$ to denote the above linear combination. Clearly, we have $P_{l,m}(p) = \alpha Q^2(p)$. Since $Q(p) \neq 0$, it follows that $\alpha = 0$. We remark that the forms in \tilde{B} vanish to order 2 at p , whereas the forms $Q_l Q$ and $Q_m Q$ vanish to order 1 at p . Therefore, by taking partial derivatives, we get

$$0 = \frac{\partial P_{l,m}}{\partial x_i}(p) = \alpha_{l,m}^{(l)} \frac{\partial Q_l Q}{\partial x_i}(p) + \alpha_{l,m}^{(m)} \frac{\partial Q_m Q}{\partial x_i}(p)$$

for $1 \leq i \leq 3$. Assume that there exists i such that $\frac{\partial Q_l Q}{\partial x_i}(p) = 0$ and $\frac{\partial Q_m Q}{\partial x_i}(p) \neq 0$. This implies $\alpha_{l,m}^{(m)} = 0$. On the other hand, since $Q_l Q$ only vanishes to order 1 at p , there exists $j \neq i$ such that $\frac{\partial Q_l Q}{\partial x_j}(p) \neq 0$. This forces $\alpha_{l,m}^{(l)} = 0$. Since, we assumed that L is linearly dependent for all pairs $(Q_l, Q_m) \in B$, the just conducted reasoning shows that the following relation holds:

$$\frac{\partial Q_l Q}{\partial x_i}(p) = 0 \Leftrightarrow \frac{\partial Q_m Q}{\partial x_i}(p) = 0 \quad \text{for all } i, l, m$$

and, hence,

$$(3.1) \quad \frac{\partial Q_l}{\partial x_i}(p) = 0 \Leftrightarrow \frac{\partial Q_m}{\partial x_i}(p) = 0 \quad \text{for all } i, l, m.$$

Recall that the basis B was constructed in the way that $\frac{\partial Q_1}{\partial x_1}(p) = 0$. Furthermore, as already remarked, due to d -independence of T , there has to exist $j \neq 1$ such that $\frac{\partial Q_j}{\partial x_1}(p) \neq 0$. Setting $l = 1$, $m = j$ and $i = 1$, this yields a contradiction to (3.1). Hence, there have to exist two forms $Q_l, Q_m \in B$ such that L is a linearly independent set. We have $|L| = \binom{d+2}{2} + 3(k+1) = \dim I_{2d}^{(2)}(\Gamma)$. It remains to consider the case $\binom{d+1}{2} + 1 \leq |\Gamma| \leq \binom{d+1}{2} + (d-2)$. It is routine to check that in this case

$$\dim I_{2d}^{(2)}(\Gamma) - \dim I_{2d}^2(\Gamma) = \left(\binom{d+2}{2} - 3|\Gamma| \right) - \left(\binom{\binom{d+2}{2} - |\Gamma| + 1}{2} \right) > 0.$$

□

From Theorem 1.6 we can immediately infer Corollary 1.7. Observe that the maximal size of a d -independent set $\Gamma \subset \mathbb{RP}^2$ is equal to $\binom{d+2}{2} - 3 = \binom{d+1}{2} + (d-2)$. Hence, Theorem 1.6 covers all d -independent sets concerning the occurrence of dimensional differences.

4. THE CASE $(n, 2d) = (4, 4)$

We now fully describe the situation with respect to $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ for sets Γ in \mathbb{R}^4 with $|\Gamma| \leq 6$. In the following, we will work with affine representatives in \mathbb{R}^4 rather than projective points in \mathbb{RP}^3 . Let $\Gamma = \{s_1, \dots, s_6\}$ be a set of six points in \mathbb{R}^4 in general linear position so that any 4 of them span \mathbb{R}^4 . We will show that Γ is 2-independent. In particular, this implies that the conditions of vanishing at $s_i \in \Gamma$ are linearly independent and therefore $\dim I_{1,2}(\Gamma) = \binom{5}{2} - 6 = 4$. It follows that the dimension of the vector space $I_4^2(\Gamma)$ spanned by squares from $I_d(\Gamma)$ is at most $\binom{5}{2} = 10$. We will show that it is equal to 10.

On the other hand, the Alexander-Hirschowitz Theorem tells us that, generically, the dimension of $I_{2,4}(\Gamma)$ is $\binom{7}{4} - 6 \cdot 4 = 11$. It is not hard to show that for 6 points in \mathbb{R}^4 in general linear position this dimension count is actually correct. Therefore, we should have a gap of 1 dimension between $P_{4,4}(\Gamma)$ and $\Sigma_{4,4}(\Gamma)$.

We now state the main result of this section, which is Theorem 1.8 from the Introduction.

Theorem 4.1. *Let $\Gamma \subset \mathbb{RP}^3$ be a finite set in general linear position. Then the following hold:*

(i) *If $|\Gamma| = 6$, then*

$$\dim I_4^2(\Gamma) = 10 < 11 = \dim I_4^{(2)}(\Gamma).$$

(ii) *If $|\Gamma| \leq 5$, then*

$$\dim I_4^2(\Gamma) = \dim I_4^{(2)}(\Gamma).$$

The proof of the above theorem will require several preparatory results. We start with a special construction for (i).

To every 3 element subset $T = \{t_1, t_2, t_3\}$ of $\{1, \dots, 6\}$, we can associate the hyperplane L_T spanned by the vectors s_{t_1} , s_{t_2} and s_{t_3} .

We want to construct a double covering of s_1, \dots, s_6 by 4 hyperplanes of the form L_T with some nice combinatorial properties. We select 4 triples T_i such that any two of them intersect in exactly one element of $\{1, \dots, 6\}$ and each element is contained in precisely two triples. Here is an example of such a covering, which is not unique:

$$T_1 = \{1, 2, 3\}, \quad T_2 = \{1, 4, 5\}, \quad T_3 = \{2, 4, 6\}, \quad T_4 = \{3, 5, 6\}.$$

To every such covering, we can associate the complementary covering, where we replace the triple T_i with its complement \overline{T}_i . So, in the given example, $\overline{T}_1 = \{4, 5, 6\}$, $\overline{T}_2 = \{2, 3, 6\}$, $\overline{T}_3 = \{1, 3, 5\}$ and $\overline{T}_4 = \{1, 2, 4\}$. We observe that the complementary covering also shares the property that any two triples intersect in exactly one point and that every point is contained in exactly two triples.

To each triple T , we associate the linear functional with kernel L_T . We can think of this functional as the inner product with the unit normal vector to L_T , which is unique up to a sign. The choice of sign will not make a difference to us. We let u_i and v_i be a unit normal vector to L_{T_i} and $L_{\overline{T}_i}$, respectively.

The vectors u_i and v_i form a pair of bases of \mathbb{R}^4 . The key is to work with the dual configurations. We define u_i^* to be vectors such that

$$\langle u_i^*, u_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

One way to think about u_i^* is that if we form the matrix U with rows u_i , then u_i^* form the columns of U^{-1} . We define vectors v_i^* in the same way for v_i .

We will show that the four forms

$$\begin{aligned} Q_1(x) &= \langle x, u_1 \rangle \langle x, v_1 \rangle, & Q_2(x) &= \langle x, u_2 \rangle \langle x, v_2 \rangle, \\ Q_3(x) &= \langle x, u_3 \rangle \langle x, v_3 \rangle, & Q_4(x) &= \langle x, u_4 \rangle \langle x, v_4 \rangle \end{aligned}$$

form a basis of $I_2(\Gamma)$. This factoring basis will allow us to prove 2-independence of Γ , and pairwise products $Q_i Q_j$ with $1 \leq i \leq j \leq 4$ will form a basis of $I_4^2(\Gamma)$.

The vectors u_i and v_i are not just two arbitrary sets of bases of \mathbb{R}^4 . Since they come from a configuration of 6 points in general linear position, they carry some structure. The following simple lemma will be crucial to our proofs.

Lemma 4.2. *For all $1 \leq i, j \leq 4$ the following hold:*

$$\langle u_i, v_j^* \rangle \neq 0 \text{ and } \langle v_i, u_j^* \rangle \neq 0.$$

Proof. By symmetry, it suffices to prove only one of the two assertions. Again, by symmetry, it will be enough to show that $\langle u_1^*, v_1 \rangle \neq 0$ and $\langle u_1^*, v_2 \rangle \neq 0$.

Suppose that $\langle u_1^*, v_1 \rangle = 0$. Then, it follows that v_1 is in the span of u_2, u_3, u_4 . Let

$$v_1 = \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4.$$

Now consider the inner product $\langle v_1, s_4 \rangle$. Recall that v_1 came from the triple $\{4, 5, 6\}$, u_2 from $\{1, 4, 5\}$, u_3 from $\{2, 4, 6\}$ and u_4 from $\{3, 5, 6\}$. It follows that

$$\langle v_1, s_4 \rangle = 0 = \alpha_4 \langle s_4, u_4 \rangle.$$

Being the points s_i in general linear position implies that $\langle s_4, u_4 \rangle \neq 0$ and, therefore, $\alpha_4 = 0$. By considering inner products of v_1 with s_5 and s_6 , we can also show that $\alpha_2 = \alpha_3 = 0$, which yields a contradiction.

Similarly, if $\langle u_1^*, v_2 \rangle = 0$, then v_2 is in the span of u_2, u_3, u_4 . Let

$$v_2 = \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4.$$

Recall that v_2 came from the triple $\{2, 3, 6\}$, u_2 from $\{1, 4, 5\}$, u_3 from $\{2, 4, 6\}$ and u_4 from $\{3, 5, 6\}$. By an analogous argument as before, we can establish that $\alpha_2 = 0$ by taking inner products with s_6 . Then, we use the inner product with s_2 to show that $\alpha_4 = 0$ and we will arrive at a contradiction. \square

Lemma 4.3. *The forms Q_i , $1 \leq i \leq 4$ form a basis of $I_2(\Gamma)$. Furthermore, the pairwise products $Q_i Q_j$ with $1 \leq i \leq j \leq 4$ form a basis of $I_4^2(\Gamma)$ and $\dim I_4^2(\Gamma) = 10$.*

Proof. It is not hard to show that $I_2(\Gamma)$ has dimension 4. To show this claim, it suffices to prove that the polynomials Q_i are linearly independent. Consider the values of Q_i at the points u_i^* .

From the definition of the dual points u_i^* and Lemma 4.2, it follows that $Q_i(u_i^*) = \langle u_i^*, v_i \rangle \neq 0$ and $Q_i(u_j^*) = 0$ if $i \neq j$. Therefore, if $P = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_3 + \alpha_4 Q_4 = 0$ for $\alpha_i \in \mathbb{R}$, then, by considering $P(u_i^*)$, we can see that $\alpha_i = 0$ for each i and, therefore, the Q_i are linearly independent.

Now consider pairwise products $Q_i Q_j$ for $1 \leq i < j \leq 4$. These forms clearly belong to $I_4^2(\Gamma)$, and we need to show their linear independence. Of all the pairwise products only Q_i^2 does not vanish at u_i^* . Therefore, the squares Q_i^2 are linearly independent from all other pairwise products and it remains to show linear independence of $Q_i Q_j$ for $1 \leq i < j \leq 4$.

By Lemma 4.2, only the products $Q_i Q_j$ vanish at u_i^* to order 1. If both indices are distinct from i , then the product vanishes to order 2. Suppose that these products are linearly dependent, i. e., there exists a linear combination such that

$$\sum_{1 \leq i < j \leq 4} \alpha_{ij} Q_i Q_j = 0,$$

where not all α_{ij} are zero. Differentiating and subsequently evaluating this equation at u_1^* , we obtain

$$\sum_{1 < j \leq 4} \alpha_{1j} \frac{\partial Q_1 Q_j}{\partial x_k}(u_1^*) = 0, \quad 1 \leq k \leq 4,$$

which is equivalent to

$$\sum_{1 < j \leq 4} \alpha_{1j} \frac{\partial Q_j}{\partial x_k}(u_1^*) = 0, \quad 1 \leq k \leq 4.$$

This is a 4×3 system of linear equations in the variables $\alpha_{12}, \alpha_{13}, \alpha_{14}$. Denote the corresponding coefficient matrix by A . Assume that this system has a nontrivial solution, meaning that all 3×3 minors of A must vanish. Considering the cross product of the three vectors $u_2, u_3, u_4 \in \mathbb{R}^4$ (see, e. g., [14]), we can see that the entries of the cross product are exactly equal to an alternating 3×3 minor of A . Hence, we conclude that the cross product is the zero vector, implying that the three vectors u_2, u_3, u_4 are linearly dependent, which is a contradiction since u_1, u_2, u_3, u_4 form a basis of \mathbb{R}^4 . Hence, we conclude that $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0$. Analogously, by following this procedure with u_2^*, u_3^* and u_4^* , we can infer that $\alpha_{23} = \alpha_{24} = \alpha_{34} = 0$ and, hence, all pairwise products are linearly independent. Since pairwise products $Q_i Q_j$ span $I_4^2(\Gamma)$ and are linearly independent, it finally follows that $\dim I_4^2(\Gamma) = 10$. \square

We are now ready to show 2-independence of Γ .

Proposition 4.4. *Let Γ be a set of 6 points in \mathbb{R}^4 in general linear position. Then Γ is 2-independent.*

Proof. We first show that Γ forces no additional zeroes on quadratic forms. Recall that $Q_i = \langle x, u_i \rangle \langle x, v_i \rangle$ and the forms Q_i form a basis of $I_2(\Gamma)$. It suffices to show that the forms Q_i have no common zeroes outside of Γ .

Let $Z_{\mathbb{C}}(Q_i)$ denote the complex zero set of the forms Q_i , $1 \leq i \leq 4$ and let z be a nonzero point in the intersection $\cap_{i=1}^4 Z_{\mathbb{C}}(Q_i)$. First assume that $z \in \cap_{i=1}^4 Z_{\mathbb{R}}(Q_i)$. It follows that, for each i , we either have $\langle z, u_i \rangle = 0$ or $\langle z, v_i \rangle = 0$. Since u_i and v_i form a basis of \mathbb{R}^4 , the vector z cannot be orthogonal to all four u_i or v_i . If $\langle z, u_i \rangle = 0$ for three indices i , which we may assume, without loss of generality, to be 1, 2 and 3, then it follows that z is a multiple of u_4^* . But then $\langle z, u_4 \rangle \neq 0$ and, from Lemma 4.2, we know that $\langle z, v_4 \rangle \neq 0$. Therefore, $Q_4(z) \neq 0$, which is a contradiction.

It thus must happen that z is orthogonal to at most two of the vectors u_i and to two of the vectors v_i . Again, without loss of generality, we may assume that z is orthogonal to u_1, u_2, v_3 and v_4 . Since u_1 comes from the triple $\{1, 2, 3\}$, u_2 comes from $\{1, 4, 5\}$, v_3 comes from $\{1, 3, 5\}$ and v_4 comes from $\{1, 2, 4\}$, it follows that z lies in the intersection of the spans of $\{s_1, s_2, s_3\}$, $\{s_1, s_4, s_5\}$, $\{s_1, s_3, s_5\}$ and $\{s_1, s_2, s_4\}$. Since the points s_i are in general linear position, we infer that s_1 spans this intersection. The other points s_i arise in the same manner from choosing different pairs of u_i 's

and v_i 's. If z is complex, then the same arguments as before applied to the real and imaginary part of z imply the claim.

For the second condition of 2-independence, we need to show that for any $s_i \in S$ there exists a unique (up to a constant multiple) form in $I_2(\Gamma)$ that is singular at s_i . Again, by symmetry, we only need to prove this for s_1 . By construction, s_1 is orthogonal to u_1, u_2, v_3 and v_4 . Therefore, it follows that $\nabla Q_1(s_1) = \langle v_1, s_1 \rangle u_1$, $\nabla Q_2(s_1) = \langle v_2, s_1 \rangle u_2$, $\nabla Q_3(s) = \langle u_3, s_1 \rangle v_3$ and $\nabla Q_4(s) = \langle u_4, s_1 \rangle v_4$. The coefficients of the vectors u_1, u_2, v_3 and v_4 are nonzero, and since the points s_i are in general linear position, it follows that u_1, u_2, v_3 and v_4 span the vector space s_1^\perp . Therefore, there is only one (up to a constant multiple) linear combination of gradients of Q_i that vanishes at s_1 . \square

Note that from the above proofs, it follows that $\dim \Sigma_{4,4}(\Gamma) = 10$. On the other hand, by the Alexander-Hirschowitz Theorem we know that $\dim P_{4,4}(\Gamma) = 11$. We have thus shown part (i) of Theorem 4.1. In the remaining part of this section, we will provide the proof of Theorem 4.1 (ii).

Proposition 4.5. *Let $\Gamma \subset \mathbb{R}^4$ be in general linear position with $|\Gamma| \leq 6$. Then Γ is 2-independent.*

Proof. For $|\Gamma| = 6$, the statement is already proven in Proposition 4.4. Let now $|\Gamma| \leq 5$. It is easy to see that the first condition of 2-independence is still satisfied whenever points are in general linear position. Indeed, we already know that the forms Q_1, \dots, Q_4 do not have any zeroes outside of $\{s_1, \dots, s_6\}$. We define $Q_5(x) = \langle x, u_1 \rangle \langle x, u_2 \rangle$, which extends Q_1, \dots, Q_4 to a basis of $I_2(\Gamma)$. Since the points are in general linear position, we have $Q_5(s_6) \neq 0$. Hence, we see that the first condition of 2-independence is satisfied for $|\Gamma| = 5$. If Γ is of smaller cardinality, we can always extend Γ to a set $\tilde{\Gamma}$ of cardinality 5 that is in general linear position. In the next step, one can extend a basis Q_1, \dots, Q_5 for $I_2(\tilde{\Gamma})$ to a basis of $I_2(\Gamma)$. Using that those new polynomials do not vanish at all of $\tilde{\Gamma}$ and using that $\tilde{\Gamma}$ satisfies the first condition of 2-independence, one can infer that also Γ satisfies this condition.

It remains to verify the second condition of 2-independence. I.e., we need to show that the vector space of quadratic forms vanishing at Γ and that are singular at exactly one point of Γ is of dimension $7 - |\Gamma|$. We proceed by induction on $|\Gamma|$. For $|\Gamma| = 6$, the claim follows from Proposition 6.3. For $|\Gamma| < 6$, we extend Γ to a set $\tilde{\Gamma}$ with $|\tilde{\Gamma}| = 6$ in general linear position. By induction, we know that the vector space of quadratic forms vanishing at $\tilde{\Gamma}$ and that are singular at one point of Γ is of dimension 1. This already implies that the dimension of the vector space of quadratic forms vanishing at Γ and that are singular at exactly one point of Γ is of dimension at most $1 + |\tilde{\Gamma} \setminus \Gamma|$. Hence, it suffices to show that, when decreasing the number of points in Γ , we gain one new linearly independent relation per point. We demonstrate this explicitly only for $|\Gamma| = 5$ since all the other cases follow the same line of arguments. Let $P = \sum_{i=1}^5 \alpha_i Q_i$ be singular at s_1 , where $\alpha_i \in \mathbb{R}$. Then $\nabla P(s_1) = \sum_{i=1}^5 \alpha_i \nabla Q_i(s_1)$. As in the proof of Proposition 4.4 (up to multiples), there is only one possible solution of this equation. Moreover, Q_5 is clearly double-vanishing at s_1 and linearly independent of the former linear combination, which shows that the dimension of the vector space of quadratic forms vanishing at s_1, \dots, s_5 and being singular at s_1 is 2. For s_2, \dots, s_5 , the situation is a bit different since $\nabla Q_5(s_i) \neq 0$ for $2 \leq i \leq 5$. If P is singular at s_2 , then

$$\nabla P(s_2) = (\alpha_1 \langle s_2, v_1 \rangle + \alpha_5 \langle s_2, u_2 \rangle) u_1 + \alpha_2 \langle s_2, u_2 \rangle v_2 + \alpha_3 \langle s_2, v_3 \rangle u_3 + \alpha_4 \langle s_2, u_4 \rangle v_4.$$

The same arguments as before show that $\alpha_1 \langle s_2, v_1 \rangle + \alpha_5 \langle s_2, u_2 \rangle, \alpha_2, \alpha_3, \alpha_4$ are uniquely determined (up to multiples). This gives one additional degree of freedom for choosing α_1 and α_5 . Hence, again, the required dimension is 2. For the other cases, one proceeds in an analogous way. \square

We remark that for $n = 4$ the above proposition is stronger than the consequence of Proposition 1.5 since it explicitly classifies which generic point configurations in \mathbb{R}^4 are 2-independent.

Consider the following forms

$$Q_5 = \langle x, u_1 \rangle \langle x, u_2 \rangle, Q_6 = \langle x, u_1 \rangle \langle x, u_3 \rangle, Q_7 = \langle x, u_2 \rangle \langle x, u_4 \rangle, Q_8 = \langle x, u_1 \rangle \langle x, u_4 \rangle.$$

Finally, we can provide the proof of Theorem 4.1 (ii).

Proof of Theorem 4.1(ii). First, consider the case $|\Gamma| = 5$. Then, $\dim I_2(\Gamma) = 5$ and the forms Q_1, \dots, Q_5 form a basis of $I_2(\Gamma)$ (to see this, evaluate at u_1^*, \dots, u_4^* and s_6). We claim that all $\binom{5+1}{2} = 15$ pairwise products $Q_i Q_j$, $1 \leq i, j \leq 5$ are linearly independent, which is the right dimension count, since $\dim I_4^{(2)}(\Gamma) = 35 - 20 = 15$. Again, by evaluating at the points u_1^*, \dots, u_4^* and s_6 , we see that the pairwise products Q_i^2 , $1 \leq i \leq 5$ are linearly independent from the pairwise products $Q_i Q_j$ with $1 \leq i < j \leq 5$. Hence, it remains to prove linear independence of the latter ones. For this aim, we use similar techniques as before. Suppose that those forms are linearly dependent, i.e., there exists a linear combination

$$(4.1) \quad \sum_{1 \leq i < j \leq 5} \alpha_{ij} Q_i Q_j = 0,$$

where not all α_{ij} are zero. Consider the evaluation of (4.1) at the point u_4^* . The only forms that vanish with order 1 at u_4^* are $Q_1 Q_4, Q_2 Q_4$ and $Q_3 Q_4$. The remaining ones vanish to higher order. Hence, differentiating P and subsequently evaluating at u_4^* yields the following 4×3 system of linear equations (note that $Q_i(u_i^*) \neq 0$):

$$\alpha_{14} \frac{\partial Q_1}{\partial x_k}(u_4^*) + \alpha_{24} \frac{\partial Q_2}{\partial x_k}(u_4^*) + \alpha_{34} \frac{\partial Q_3}{\partial x_k}(u_4^*) = 0, \quad 1 \leq k \leq 4.$$

By the same arguments as in the proof of Lemma 4.3, we can conclude that $\alpha_{14} = \alpha_{24} = \alpha_{34} = 0$. Now evaluate (4.1) at u_3^* . This time, the forms $Q_1 Q_3, Q_2 Q_3$ are the only pairwise products vanishing with order 1. This yields a 4×2 system of linear equations, from which we infer $\alpha_{13} = \alpha_{23} = 0$ since, in this case, vanishing of all 2×2 minors implies linear dependence of u_1, u_2 contradicting the fact that u_1, \dots, u_4 form a basis of \mathbb{R}^4 . Analogously, evaluating (4.1) at u_1^* yields $\alpha_{12} = \alpha_{15} = 0$ by exactly the same arguments as before. So, we are left with the pairwise products $Q_2 Q_5, Q_3 Q_5, Q_4 Q_5$. These forms are clearly linearly independent since the forms Q_i , $2 \leq i \leq 4$ are linearly independent. Hence, the claim follows.

Now, assume $|\Gamma| < 5$. In this case, note that there is always an overcount in the pairwise products. For example, in the case $|\Gamma| = 4$, there are $\binom{6+1}{2} = 21$ pairwise products. Since $\dim I_4^{(2)}(\Gamma) = 35 - 16 = 19$, we need to prove that, out of these 21 pairwise products, there exist 19 pairwise products that are linearly independent. Since the proof uses exactly the same strategy as in the case $|\Gamma| = 5$ and does not contain new arguments, in the next table, we only provide a basis for $I_4^2(\Gamma)$ for $|\Gamma| < 5$. For $|\Gamma| = m$, the forms Q_i , $1 \leq i \leq 10 - m$ form a basis of $I_2(\Gamma)$. We set $L = \{Q_i Q_j | i \leq j\}$ and use $B(\Gamma) \subset L$ to denote a basis of $I_4^2(\Gamma)$.

$ \Gamma $	$\dim I_4^2(\Gamma)$	$\dim I_4^{(2)}(\Gamma)$	$B(\Gamma)$
4	19	19	$L \setminus \{Q_2 Q_6, Q_3 Q_5\}$
3	23	23	$L \setminus \{Q_1 Q_7, Q_2 Q_6, Q_3 Q_5, Q_4 Q_5, Q_5 Q_6\}$
2	27	27	$L \setminus \{Q_1 Q_7, Q_2 Q_6, Q_2 Q_8, Q_3 Q_8, Q_4 Q_6, Q_5 Q_7, Q_6 Q_7, Q_6 Q_8, Q_7 Q_8\}$

Note that, for $|\Gamma| = 1$, we always have $\dim I_4^2(\Gamma) = \dim I_4^{(2)}(\Gamma)$, and, hence, the proof is finished. \square

From Theorem 4.1 we can immediately infer Corollary 1.9.

5. EXPLICIT CHARACTERIZATION OF 1-DIMENSIONAL DIFFERENCES

The aim of this section is to explicitly characterize a dimensional difference between $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ for $(n, 2d) = (4, 4)$ and $(n, 2d) = (3, 6)$. By Theorems 1.8 and 1.6, the first time

those differences occur is exactly for $|\Gamma| = 6$, respectively, $|\Gamma| = 7$, and it follows by dimension counting that the dimensional difference in these cases is exactly one. Hence, constructing forms in $I_{2d}^{(2)}(\Gamma) \setminus I_{2d}^2(\Gamma)$ already yields a complete characterization of the occurring dimensional differences in the two smallest cases, in which nonnegative forms that are not sums of squares exist.

5.1. The case $(n, 2d) = (4, 4)$. In order to describe the dimensional difference, we need to construct a form R of degree 4 such that $R \in I_4^{(2)}(\Gamma) \setminus I_4^2(\Gamma)$.

Proposition 5.1. *Let $\Gamma \subset \mathbb{R}^4$ be in general linear position with $|\Gamma| = 6$. Set $R = \langle x, u_1 \rangle \langle x, u_2 \rangle \langle x, u_3 \rangle \langle x, u_4 \rangle$. Then $R \in I_4^{(2)}(\Gamma) \setminus I_4^2(\Gamma)$.*

Proof. We know that products $Q_i Q_j$ with $1 \leq i \leq j \leq 4$ form a basis of $I_4^2(\Gamma)$. We observe that $R(u_i^*) = 0$ for all i , and the only form from the spanning set that does not vanish at u_i^* is Q_i^2 . Therefore, if we assume that R is spanned by $Q_i Q_j$, then R needs to be spanned by products $Q_i Q_j$ with $i \neq j$.

Now consider $R(v_k^*)$. By Lemma 4.2, we know that $R(v_k^*) \neq 0$. However, $Q_i Q_j(v_k^*) = 0$ since $\langle v_i^*, v_k \rangle = 0$ for $i \neq k$. Therefore, we arrive at a contradiction. \square

Corollary 5.2. *Let $\Gamma \subset \mathbb{R}^4$ be in general linear position with $|\Gamma| = 6$. There exists $p \in P_{4,4}(\Gamma) \setminus \Sigma_{4,4}(\Gamma)$ with $\Gamma \subset Z(p)$. These forms can be constructed via $Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + \varepsilon R$ for sufficiently small $\varepsilon > 0$.*

We now provide an explicit example for a form as described in the above corollary.

Example 1. Let $s_1 = (0, 0, 1, 1)$, $s_2 = (0, 1, 0, 1)$, $s_3 = (0, 1, 1, 0)$, $s_4 = (1, 0, 0, 1)$, $s_5 = (1, 0, 1, 0)$ and $s_6 = (1, 1, 0, 0)$ and $\Gamma = \{s_1, \dots, s_6\}$. The following polynomials form a basis of $I_2(\Gamma)$.

$$\begin{aligned} Q_1(x) &= x_1(x_1 - x_2 - x_3 - x_4), & Q_2(x) &= x_2(x_2 - x_1 - x_3 - x_4), \\ Q_3(x) &= x_3(x_3 - x_1 - x_2 - x_4), & Q_4(x) &= x_4(x_4 - x_1 - x_2 - x_3) \end{aligned}$$

The form $R = \langle x, u_1 \rangle \langle x, u_2 \rangle \langle x, u_3 \rangle \langle x, u_4 \rangle$ from Proposition 5.1 becomes

$$R = \langle x, e_1 \rangle \langle x, e_2 \rangle \langle x, e_3 \rangle \langle x, e_4 \rangle = x_1 x_2 x_3 x_4.$$

One can verify that $Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 + R \in P_{4,4}(\Gamma) \setminus \Sigma_{4,4}(\Gamma)$.

5.2. The case $(n, 2d) = (3, 6)$. We consider the case $(n, 2d) = (3, 6)$ and $\Gamma \subset \mathbb{R}^3$ with $|\Gamma| = 7$. Compared to the case $(n, 2d) = (4, 4)$ from the previous section, the situation becomes more involved. Let $\Gamma = \{s_1, \dots, s_7\}$. Let u_1, u_2, u_3 be the normal vectors to the hyperplanes passing through s_1, s_2 , respectively, s_3, s_4 , respectively, s_5, s_6 . Note that, generically, u_1, u_2, u_3 are a basis of \mathbb{R}^3 . Let u_1^*, u_2^*, u_3^* be the dual basis to u_1, u_2, u_3 . Furthermore, we define K_1, K_2, K_3 to be the conics passing through the points s_i with $i \in \{3, 4, 5, 6, 7\}$, respectively $i \in \{1, 2, 5, 6, 7\}$, respectively $i \in \{1, 2, 3, 4, 7\}$. Generically, we can assume that $K_i(u_j^*) \neq 0$ for $1 \leq i \neq j \leq 3$.

Lemma 5.3. *Let $Q_1(x) = \langle x, u_1 \rangle K_1$, $Q_2(x) = \langle x, u_2 \rangle K_2$ and $Q_3(x) = \langle x, u_1 \rangle K_3$. Then $\{Q_1(x), Q_2(x), Q_3(x)\}$ is a basis of $I_3(\Gamma)$.*

Proof. Assume that $Q_1(x), Q_2(x)$ and $Q_3(x)$ are linearly dependent, i.e., there exist $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ such that

$$(5.1) \quad 0 = \alpha_1 Q_1(x) + \alpha_2 Q_2(x) + \alpha_3 Q_3(x).$$

Evaluating (5.1) at u_i^* and using that $\langle u_i^*, u_j \rangle = 0$, for $i \neq j$, and $K_i(u_j^*) \neq 0$, for all $1 \leq i, j \leq 3$, we infer $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$. \square

We now construct an explicit form $R \in I_6^{(2)}(\Gamma) \setminus I_6^2(\Gamma)$.

Proposition 5.4. *Let $R = K\langle x, u_1 \rangle \langle x, u_2 \rangle \langle x, u_3 \rangle$, where K is the unique cubic double vanishing at the point s_7 and vanishing at s_1, \dots, s_6 with multiplicity 1 such that $K(u_i^*) \neq 0$ for $1 \leq i \leq 3$. Then $R \in I_6^{(2)}(\Gamma) \setminus I_6^2(\Gamma)$.*

Proof. By construction, it is clear that $R \in I_6^{(2)}(\Gamma)$. We have to prove that R together with all pairwise products $Q_i Q_j$, $1 \leq i \leq j \leq 3$ form a linearly independent set of polynomials. Suppose these forms were linearly dependent, i.e., there exists a nontrivial linear combination $\alpha_R R + \sum_{1 \leq i \leq j \leq 3} \alpha_{ij} Q_i Q_j = 0$. By evaluating this relation at u_i^* , we get $\alpha_{ii} = 0$ for $1 \leq i \leq 3$. Hence, the squares are linearly independent from the pairwise products $Q_i Q_j$ with $1 \leq i < j \leq 3$ and it remains to prove linear independence of the forms $Q_i Q_j$ with $1 \leq i < j \leq 3$. Suppose that these forms are linearly dependent. The forms $Q_1 Q_2$ and $Q_1 Q_3$ vanish to order 1 at u_1^* , whereas the forms $Q_2 Q_3$ and R vanish to order 2. Hence, by taking the partial derivatives and subsequently evaluating them at u_1^* , we get for $k \in \{2, 3\}$

$$\alpha_{1k} \frac{\partial Q_1 Q_k}{\partial x_j}(u_1^*) = \alpha_{1k} \left(\frac{\partial Q_1}{\partial x_j} \cdot Q_k + Q_1 \cdot \frac{\partial Q_k}{\partial x_j} \right)(u_1^*) = \alpha_{1k} Q_1(u_1^*) \cdot \frac{\partial Q_k}{\partial x_j}(u_1^*) = 0, \quad 1 \leq j \leq 3$$

and hereby the following 3×2 system of linear equations:

$$\alpha_{12} Q_1(u_1^*) \cdot \frac{\partial Q_2}{\partial x_j}(u_1^*) + \alpha_{13} Q_1(u_1^*) \cdot \frac{\partial Q_3}{\partial x_j}(u_1^*) = 0, \quad 1 \leq j \leq 3.$$

Suppose that this system has a nontrivial solution. This is the case if and only if the rank of the corresponding coefficient matrix is one. Hence, all 2×2 minors must vanish. One can check (by taking the partial derivatives and considering $K_i(u_j^*) \neq 0$) that the rank is one if and only if the cross product of the two vectors u_2 and u_3 is zero, implying that u_2 and u_3 are linearly dependent. But this is a contradiction since u_1, u_2 and u_3 form a basis of \mathbb{R}^3 . Hence, the above system can only have the trivial solution $\alpha_{12} = \alpha_{13} = 0$, and we are now left with the equation $\alpha_{23} Q_2 Q_3 + \alpha_R R = 0$. However, since the form $Q_2 Q_3$ vanishes to order 1 at u_2^* and the form R with order 2 at u_2^* , we get $\alpha_{23} = 0$ by taking the partial derivatives and, hence, $\alpha_R = 0$, which finishes the proof. \square

Corollary 5.5. *Let $\Gamma \subset \mathbb{R}^3$ be 3-independent and $|\Gamma| = 7$. Under the assumptions of the previous lemma there exists $p \in P_{3,6}(\Gamma) \setminus \Sigma_{3,6}(\Gamma)$ with $\Gamma \subset Z(p)$. These forms can be constructed via $Q_1^2 + Q_2^2 + Q_3^2 + \varepsilon R$ for sufficiently small $\varepsilon > 0$.*

Example 2. *Note that the condition $K_i(u_j^*) \neq 0$ is essential. Let $\Gamma = \{s_1, \dots, s_7\}$ with*

$$s_1 = (1, 0, 0), s_2 = (0, 1, 0), s_3 = (0, 0, 1), s_4 = (1, 1, 0), s_5 = (1, 0, 1), s_6 = (0, 1, 1), s_7 = (1, 1, 1).$$

It can be checked that this set of points is 3-independent. However, one can verify that $u_3^ = (\frac{1}{2}, \frac{1}{2}, 0)$ and hence it represents the same projective point as s_4 . In particular, we always have $K_1(u_3^*) = 0$ by construction. However, we can perturb the point s_4 to $\tilde{s}_4 = (1, -2, 2)$. The new set of points remains 3-independent, and we have $K_i(u_j^*) \neq 0$ for $1 \leq i \leq j \leq 3$. The three basis polynomials are then given by*

$$\begin{aligned} Q_1(x_1, x_2, x_3) &= (3x_1x_2 - x_1x_3 - 2x_2x_3)(-x_3 + x_2 + x_1), \\ Q_2(x_1, x_2, x_3) &= (x_2 - x_3)(x_1 - x_3)(2x_1 + x_2), \\ Q_3(x_1, x_2, x_3) &= x_3(8x_1^2 + x_2^2 - 8x_1x_3 - x_2x_3). \end{aligned}$$

Furthermore, we have

$$R = -x_3(2x_1 + x_2)^2(x_1 + x_2 - x_3)(x_2 - x_3)(-x_3 + x_1).$$

One can check that

$$Q_1^2 + Q_2^2 + Q_3^2 + R \in P_{3,6}(\Gamma) \setminus \Sigma_{3,6}(\Gamma).$$

6. GENERAL (NAIVE) BOUNDS FOR DIMENSIONAL DIFFERENCES BETWEEN $P_{n,2d}(\Gamma)$ AND $\Sigma_{n,2d}(\Gamma)$

We now want to derive some naive dimension counts for the dimensions of $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$, which help to understand when, theoretically, dimensional gaps between these faces can occur. Moreover, these counts yield some a priori bounds on the minimal size of Γ such that dimensional gaps between these faces can be observed. A first step in this direction is the following lemma.

Lemma 6.1. *Let Γ be a d -independent set of k points in \mathbb{RP}^{n-1} . Then $\dim P_{n,2d}(\Gamma) \geq \binom{n+2d-1}{2d} - kn$ and $\dim \Sigma_{n,2d}(\Gamma) \leq \binom{\binom{n+d-1}{d}-k+1}{2}$.*

Proof. The dimension of $I_{2d}^{(2)}(\Gamma)$ is at least $\binom{n+2d-1}{2d} - kn$ since we are imposing at most kn linearly independent conditions by forcing forms to double vanish at all points of Γ . From Proposition 1.3 we know that $P_{n,2d}(\Gamma)$ is full-dimensional in $I_{2d}^{(2)}(\Gamma)$ and, thus, the bound for the dimension of $P_{n,2d}(\Gamma)$ follows.

Since Γ is d -independent, we know that the dimension of $I_d(\Gamma)$ is $\binom{n+d-1}{d} - k$. We can have at most $\binom{\dim I_d(\Gamma)+1}{2}$ linearly independent pairwise products coming from $I_d(\Gamma)$ and, therefore, the dimension of $I_{2d}^2(\Gamma)$ is at most $\binom{\binom{n+d-1}{d}-k+1}{2}$. Since the containment $\Sigma_{n,2d}(\Gamma) \subset I_{2d}^2(\Gamma)$ is full-dimensional by Proposition 1.1, the bound for $\Sigma_{n,2d}(\Gamma)$ follows. \square

For a d -independent set Γ of size k let $G_{n,2d}(k)$ be the size of the minimal gap between the dimensions of $P_{n,2d}(\Gamma)$ and $\Sigma_{n,2d}(\Gamma)$ which by Lemma 6.1 is given by:

$$(6.1) \quad G_{n,2d}(k) = \binom{n+2d-1}{2d} - kn - \binom{\binom{n+d-1}{d} - k + 1}{2}.$$

From Section 2.2 we know that there exist d -independent sets of any cardinality $k \leq \binom{n+d-1}{d} - n$. We want to determine the smallest positive integer k for which $G_{n,2d}(k) > 0$, and we want to find the maximum of $G_{n,2d}(k)$.

Proposition 6.2. *The function $G_{n,2d}(k)$ is maximized at $k = \binom{n+d-1}{d} - n$. Its value and the largest gap are*

$$(6.2) \quad \binom{n+2d-1}{2d} - n \binom{n+d-1}{d} + \binom{n}{2}.$$

The smallest value of k such that $G_{n,2d}(k) > 0$ is the smallest integer strictly greater than:

$$(6.3) \quad \binom{n+d-1}{d} - n + \frac{1}{2} - \sqrt{\left(n - \frac{1}{2}\right)^2 + 2 \binom{n+2d-1}{2d} - 2n \binom{n+d-1}{d}}.$$

Proof. We observe that $G_{n,2d}(k)$ is a quadratic function of k with a negative leading coefficient. It is easy to show that $G_{n,2d}(k)$ attains its maximum value at $k = \binom{n+d-1}{d} - n + \frac{1}{2}$. Therefore, the maximum value of $G_{n,2d}(k)$ for an integer k will occur with $k = \binom{n+d-1}{d} - n$ and it is a matter of easy simplification to obtain equation (6.2).

The bound in equation (6.3) comes from simply calculating the smallest root of $G_{n,2d}(k)$. We skip the routine application of the quadratic formula. \square

We make several remarks. First we observe that the largest gap number from Proposition 6.2

$$\binom{n+2d-1}{2d} - n \binom{n+d-1}{d} + \binom{n}{2}$$

is zero in all cases, where the cones $P_{n,2d}$ and $\Sigma_{n,2d}$ coincide. However, this number is strictly positive in the cases, where there are nonnegative forms that are not sums of squares. In the smallest cases $n = 4$, $2d = 4$ and $n = 3$, $2d = 6$, in which $P_{n,2d}$ is strictly larger than $\Sigma_{n,2d}$, the gap number is 1.

However, as either n or d grows, we can see that the dimensional gap between exposed faces of $P_{n,2d}$ and $\Sigma_{n,2d}$ grows and, asymptotically, it approaches the full dimension of the vector space $P_{n,2d}$.

We note that the bound from Equation 6.3 simplifies remarkably for $n = 3$. In this case, we get the bound of $\binom{d+2}{2} - d - 1$, and we need to take the smallest integer above that, which leads to

$$k = \binom{d+2}{2} - d = \binom{d+1}{2} + 1.$$

This is actually the correct bound for the case of $n = 3$ as we proved in Theorem 1.6.

Though, in general, for $n \geq 4$ the formula does not appear to simplify and the bound given is not going to be optimal, Theorem 4.1 implies optimality of the naive dimension count also in the case $(n, 2d) = (4, 4)$. The non-optimality in the general case is caused by an overcount in the bound for the dimension of the vector space $I_{2d}^2(\Gamma)$.

We note that for $k = \binom{n+d-1}{d} - n$, which leads to the largest gap, the bound on the dimension of $I_{2d}^2(\Gamma)$ is also optimal, generically. We can see this from the example of the d -independent set $S_{n,d}$ from Section 2.2, which has exactly this cardinality. Indeed, for $S_{n,d}$, it is not hard to show that all pairwise products of the forms Q_i , which form the basis of $I_d(S_{n,d})$, are linearly independent in $H_{n,2d}$. This shows that the dimension of $I_{2d}^2(S_{n,d})$ is $\binom{n+1}{2}$, which is exactly equal to the bound we use.

Question 6.1. *From the above discussion it is natural to ask the following questions.*

- (i) *In addition to the known cases $((n, 2d) \in \{(3, 2d), (4, 4)\})$, are there other cases, in which the naive bound from Proposition 6.2 for the smallest cardinality of a set Γ forcing a dimensional gap between the corresponding faces is correct?*
- (ii) *Given n, d , is it possible to characterize the set of integers S such that $\dim P_{n,2d}(\Gamma) > \dim \Sigma_{n,2d}(\Gamma)$ if and only if $|\Gamma| \in S$?*
- (iii) *Can the naive bounds from Proposition 6.2 be further improved?*
- (iv) *Is it possible to characterize more cases for which $\dim P_{n,2d}(\Gamma) - \dim \Sigma_{n,2d}(\Gamma) = 1$?*

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